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Local 2-geodesic transitivity and clique graphs ☆

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ABSTRACT

A 2-geodesic in a graph is a vertex triple (u, v, w) such that v is adjacent to both u and w and u, w are not adjacent. We study non-complete graphs Γ in which, for each vertex u, all 2-geodesics with initial vertex u are equivalent under the subgroup of graph automorphisms fixing u. We call such graphs locally 2-geodesic transitive, and show that the subgraph $[\Gamma(u)]$ induced on the set of vertices of Γ adjacent to u is either (i) a connected graph of diameter 2, or (ii) a union mK_r of $m \geqslant 2$ copies of a complete graph K_r with $r \geqslant 1$. This suggests studying locally 2-geodesic transitive graphs according to the structure of the subgraphs $[\Gamma(u)]$. We investigate the family $\mathcal{F}(m, r)$ of connected graphs Γ such that $[\Gamma(u)] \cong mK_r$ for each vertex u, and for fixed $m \geqslant 2$, $r \geqslant 1$. We show that each $\Gamma \in \mathcal{F}(m, r)$ is the point graph of a partial linear space $\mathcal{S}$ of order $(m, r+1)$ which has no triangles (and 2-geodesic transitivity of Γ corresponds to natural strong symmetry properties of $\mathcal{S}$). Conversely, each $\mathcal{S}$ with these properties has point graph in $\mathcal{F}(m, r)$, and a natural duality on partial linear spaces induces a bijection $\mathcal{F}(m, r) \mapsto \mathcal{F}(r+1, m-1)$.

1. Introduction

In this paper, graphs are finite, simple and undirected. For a graph Γ, a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$, and a 2-*geodesic* if in addition u, w are not adjacent. A connected non-complete graph Γ with subgroup G of automorphisms is said to be *locally* $(G, 2)$-*arc transitive* or *locally* $(G, 2)$-*geodesic transitive* if, for every vertex u, the stabilizer G_u of u is transitive on the set $\Gamma(u)$ of vertices adjacent to u, and on the 2-arcs or 2-geodesics starting at u,

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respectively. (See Remark 2.1 for a comment on these definitions.) Clearly, every 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. Thus the family of non-complete locally $(G,2)$-arc transitive graphs is properly contained in the family of locally $(G,2)$-geodesic transitive graphs. If a non-complete graph Γ is locally $(G,2)$-arc transitive, then the subgraph $[\Gamma(u)]$ induced on $\Gamma(u)$ is an empty graph (that is, with no edges), for each vertex u. However, there are many locally $(G,2)$-geodesic transitive graphs whose neighborhood subgraphs are not empty graphs. One of the aims of this paper is to determine the possible structures of $[\Gamma(u)]$ for connected locally $(G,2)$-geodesic transitive graphs. Our first result shows that there are three broad categories of locally $(G,2)$-geodesic transitive graphs.

Theorem 1.1. *Let Γ be a connected non-complete locally $(G,2)$-geodesic transitive graph. Then one of the following holds.*

(1) *Γ is G-vertex transitive,* $\mathrm{girth}(\Gamma)=3$, *$[\Gamma(u)]$ is connected of diameter 2 for any vertex u of Γ, and the induced action of G_u on $\Gamma(u)$ is transitive on both vertices and ordered pairs of non-adjacent vertices.*
(2) *Γ is G-vertex transitive, and there exist integers $m\geqslant 2, r\geqslant 1$ such that for any vertex u of Γ, $[\Gamma(u)]\cong mK_r$.*
(3) *Γ is not G-vertex transitive, Γ is bipartite with biparts Δ_1 and Δ_2, and there exist positive integers m_1, m_2 with $\max\{m_1,m_2\}\geqslant 2$ such that for any $u_i\in\Delta_i$, $[\Gamma(u_i)]\cong m_iK_1$.*

Remark 1.2. If a connected non-complete locally $(G,2)$-geodesic transitive graph Γ satisfies case (2) with $r=1$, or case (3) of Theorem 1.1, then Γ contains no cycles of length 3, and hence each 2-arc is a 2-geodesic. It follows that Γ is locally $(G,2)$-arc transitive. Such graphs have been studied extensively, see [1,4–6,9–11]. We are interested in the case where Γ is not $(G,2)$-arc transitive, that is, case (1) and case (2) with $r\geqslant 2$.

In cases (1) and (2) of Theorem 1.1, Γ is G-vertex transitive, and we say that Γ is $(G,2)$-*geodesic transitive*. Theorem 1.1 shows that the study of $(G,2)$-geodesic transitive graphs falls naturally into two cases, according to whether $[\Gamma(u)]$ is connected or disconnected. We explore the latter case in this paper and will study the connected case in forthcoming work.

Definition 1.3. For positive integers m and r, we denote by $\mathcal{F}(m,r)$ the family of connected graphs Γ of valency mr such that for every vertex u, $[\Gamma(u)]\cong mK_r$, the disjoint union of m copies of the complete graph K_r.

For any pair of positive integers m, r, there exist graphs in the family $\mathcal{F}(m,r)$. For instance, the Hamming graph $\mathrm{H}(m,r+1)$ (with vertex set $Z_{r+1}^m = Z_{r+1}\times Z_{r+1}\times\cdots\times Z_{r+1}$ where $Z_{r+1}=\{0,1,\ldots,r\}$ is the ring of integers modulo $r+1$, and two vertices u,v are adjacent if and only if $u-v$ has exactly one non-zero entry) is in $\mathcal{F}(m,r)$, and it is also 2-geodesic transitive, see [3]. Hiraki, Nomura and Suzuki [8] have studied the family $\mathcal{F}(3,2)$, classifying the distance regular members. We explore a link between graphs in $\mathcal{F}(m,r)$ and a certain family of partial linear spaces, and we also find a bijection between $\mathcal{F}(m,r)$ and $\mathcal{F}(r+1,m-1)$.

A *partial linear space* $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathcal{I})$ of *order* (m,n) (where $m\geqslant 2, n\geqslant 2$) consists of a set $\mathcal{P}$ of *points*, a set $\mathcal{L}$ of *lines*, and an *incidence relation* $\mathcal{I}\subseteq\mathcal{P}\times\mathcal{L}$ such that each pair of points is incident with at most one line, each point is incident with m lines, and each line is incident with n points. The $\mathcal{S}$-*point graph* is the graph with vertex set $\mathcal{P}$ such that two points are adjacent if and only if they are incident with a common line. A *triangle* of $\mathcal{S}$ is a clique of the $\mathcal{S}$-point graph of size three such that the three points are not incident with a common line.

A *clique* of a graph Γ is a complete subgraph and a *maximal clique* is a clique which is not contained in a larger clique. The *clique graph* $C(\Gamma)$ of Γ is the graph with vertex set {all maximal cliques of Γ}, and two maximal cliques are adjacent in $C(\Gamma)$ if and only if they have at least one common vertex in Γ. The relevance of these concepts is clear in our second main result, which will be proved in Section 4.2.

Theorem 1.4. *Let Γ be a connected graph and $m \geqslant 2$, $r \geqslant 1$ be integers. Then the following two statements hold.*

(1) *$\Gamma \in \mathcal{F}(m,r)$ if and only if Γ is the $\mathcal{S}$-point graph of a partial linear space $\mathcal{S}$ of order $(m, r+1)$ which has no triangles.*
(2) *If $\Gamma \in \mathcal{F}(m,r)$, then the map $\phi_{m,r} : \Gamma \mapsto C(\Gamma)$ is a bijection from $\mathcal{F}(m,r)$ to $\mathcal{F}(r+1, m-1)$ and $\phi_{m,r}^{-1} = \phi_{r+1,m-1}$ in the sense that $\phi_{r+1,m-1}(\phi_{m,r}(\Gamma)) \cong \Gamma$. In particular, $C(C(\Gamma)) \cong \Gamma$.*

Theorem 1.4(1) provides a useful method for constructing graphs in $\mathcal{F}(m,r)$ for certain integers $m \geqslant 2, r \geqslant 1$. For example, the Tutte 8-cage is the incidence graph of a partial linear space $\mathcal{S}$ of order $(3,3)$ which has no triangles, so the $\mathcal{S}$-point graph is in $\mathcal{F}(3,2)$. Indeed this is one of the graphs characterized in [8]. Further, transitivity on 2-geodesics of a graph $\Gamma \in \mathcal{F}(m,r)$ corresponds to transitivity on the triples of points (a,b,c) of $\mathcal{S}$ such that there exist two lines $\ell_1 \neq \ell_2$, and (a,ℓ_1), (b,ℓ_1), (b,ℓ_2), $(c,\ell_2) \in \mathcal{I}$. Partial linear spaces with these transitivity properties will be studied in forthcoming work of the authors.

The *line graph* $L(\Gamma)$ of a graph Γ has the set of edges of Γ as its vertex set, and two edges are adjacent in $L(\Gamma)$ if and only if they have a common vertex in Γ. Line graphs occur in some of the $\mathcal{F}(m,r)$, see Remark 1.5 and Corollary 1.6.

Remark 1.5. (a) The only families $\mathcal{F}(m,r)$ not covered by Theorem 1.4 are those with $m=1$: the family $\mathcal{F}(1,r) = \{K_{r+1}\}$ and $C(K_{r+1}) = K_1$.

(b) If $\Gamma \in \mathcal{F}(m,1)$ with $m \geqslant 2$, then the maximal cliques have size 2 and girth$(\Gamma) \geqslant 4$. In this case, $C(\Gamma) = L(\Gamma)$ is the line graph of Γ, and by Theorem 1.4(2), $C(L(\Gamma)) \cong \Gamma$. In fact, for $\Gamma \in \mathcal{F}(m,r)$, $C(\Gamma) \cong L(\Gamma)$ if and only if $r=1$ (see Corollary 4.2).

For a positive integer n, we denote by $K_{1,n}$ the complete bipartite graph with biparts of sizes 1 and n. The following corollary is proved in Section 4.2.

Corollary 1.6. *Let $\Gamma \in \mathcal{F}(m,r)$ where $m \geqslant 1, r \geqslant 1$. Then Γ is a line graph if and only if one of the following holds*:

(1) *$m=1$, $r \neq 2$ and $\Gamma \cong K_{r+1} = L(\Sigma)$ where $\Sigma \cong K_{1,r+1}$;*
(2) *$m=1$, $r=2$ and $\Gamma \cong K_3 = L(\Sigma)$ where $\Sigma \cong K_{1,3}$ or K_3;*
(3) *$m=2$ and $\Gamma \cong L(C(\Gamma))$.*

2. Notation and concepts

In this section we give some notation and definitions which will be used in the paper. Let Γ be a graph. We use $V(\Gamma)$, $E(\Gamma)$, and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set*, *edge set* and *automorphism group*, respectively. The graph Γ is said to be *vertex transitive* if the action of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$ is transitive.

A subgraph X of Γ is an *induced subgraph* provided two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we denote by $[U]_\Gamma$ (or simply $[U]$ when there is no ambiguity) the subgraph of Γ induced by U.

For two vertices u and v in $V(\Gamma)$, the smallest integer n such that there is a path of length n from u to v is called the *distance* from u to v and is denoted by $d_\Gamma(u,v)$. If Γ is disconnected, we define $d_\Gamma(u,v) = \infty$ whenever u and v belong to different connected components of Γ. If Γ is connected, the *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximum distance $d_\Gamma(u,v)$ between vertices $u, v \in V(\Gamma)$. For a vertex $u \in V(\Gamma)$, we set $\Gamma_2(u) = \{v \in V(\Gamma) \mid d_\Gamma(u,v) = 2\}$.

In [2], the authors classified 2-geodesic transitive graphs of girth 3 and valency 4. Remark 2.1 comments on the definition of local $(G,2)$-geodesic transitivity.

Remark 2.1. For a graph Γ and $G \leqslant \mathrm{Aut}(\Gamma)$ if, for every $u \in V(\Gamma)$, G_u is transitive on 2-geodesics starting from u, it is possible that there exists a vertex v such that G_v is not transitive on arcs

Fig. 1. Two examples pertaining to Remark 2.1.

starting from v. The two graphs in Fig. 1 are examples. Thus in the definition of local $(G,2)$-geodesic transitivity, it is necessary to require that for every $u \in V(\Gamma)$, G_u is transitive on both arcs and 2-geodesics starting from u.

For a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ of order (m,n) as defined in Section 1, if $(p,\ell) \in \mathcal{I}$, then we say that the point p and the line ℓ are *incident*. The $\mathcal{S}$*-line graph* ($\mathcal{S}$*-point graph*) of $\mathcal{S}$ is the graph with vertex set $\mathcal{L}$ (vertex set $\mathcal{P}$) and two vertices are adjacent if and only if they are incident with a common point (line, respectively). We refer to the $\mathcal{S}$-line graph rather than simply the line graph to distinguish this notion from the line graph of Remark 1.5. The *incidence graph* $\overline{\mathcal{S}}$ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and edge set $\{p,\ell\}$ where $(p,\ell) \in \mathcal{I}$. We denote the "transposed" incidence relation by $\mathcal{I}^* = \{(\ell,p) \mid (p,\ell) \in \mathcal{I}\}$. It is not difficult to verify that $\mathcal{S}^* = (\mathcal{L}, \mathcal{P}, \mathcal{I}^*)$ is a partial linear space of order (n,m) and is called the *dual partial linear space* of $\mathcal{S}$. Note that $\mathcal{S}^{**} = \mathcal{S}$, and the $\mathcal{S}^*$-line graph is the $\mathcal{S}$-point graph, etc. For two partial linear spaces $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$, a bijection $\phi : \mathcal{P}_1 \mapsto \mathcal{P}_2$, $\mathcal{L}_1 \mapsto \mathcal{L}_2$ is called an *isomorphism* between $\mathcal{S}_1$ and $\mathcal{S}_2$ if, $(p_1,\ell_1) \in \mathcal{I}_1$ if and only if $(p_1,\ell_1)^\phi \in \mathcal{I}_2$.

3. Proof of Theorem 1.1

Let Γ be a connected non-complete locally $(G,2)$-geodesic transitive graph. Then, for each $u \in V(\Gamma)$, G_u is transitive on $\Gamma(u)$, and hence G is transitive on $E(\Gamma)$. If Γ is not G-vertex transitive, then Γ is bipartite and its two biparts Δ_1 and Δ_2 are the G-orbits in $V(\Gamma)$, see for example [5, Lemma 3.1] (note that in [5, Lemma 3.1], the condition that all vertices have valency at least 2 is not necessary, as if one vertex has valency 1, then $\Gamma \cong K_{1,r}$ for some $r \geqslant 1$). Thus there exist m_1, m_2 such that for any $u_i \in \Delta_i$ we have $[\Gamma(u_i)] \cong m_i K_1$, and $\max\{m_1,m_2\} \geqslant 2$ since Γ contains a 2-geodesic, so (3) holds.

Now we assume that Γ is G-vertex transitive. Then Γ is $(G,2)$-geodesic transitive. In particular, Γ is regular with valency $n \geqslant 2$ since Γ contains 2-geodesics. Let (u,v) be an arc of Γ. Then $G_{u,v}$ is transitive on $\Gamma(u) \cap \Gamma_2(v)$, and this implies that all vertices of the latter set are at the same distance i from v in $\Sigma := [\Gamma(u)]$, and with $i = 2$ when Σ is connected and $i = \infty$ otherwise.

Suppose that Σ is connected and note that G_u is transitive on $V(\Sigma)$. Since Γ is non-complete, also Σ is non-complete, and so $\mathrm{diam}(\Sigma) \geqslant 2$. Since $\Gamma(u) \cap \Gamma_2(v) = \Gamma(u) \setminus (\{v\} \cup (\Gamma(u) \cap \Gamma(v)))$ and all vertices of $\Gamma(u) \cap \Gamma_2(v)$ are at the same distance 2 from v in Σ, it follows that $\mathrm{diam}(\Sigma) = 2$. Moreover, since G_u is transitive on $V(\Sigma)$, this implies that G_u is transitive on ordered pairs of vertices at distance 2 in Σ. Thus (1) holds.

Finally, suppose that Σ is not connected and has m ($\geqslant 2$) connected components. If $\mathrm{girth}(\Gamma) \geqslant 4$, then $[\Gamma(u)] \cong nK_1$, and (2) holds. Thus we may assume that $\mathrm{girth}(\Gamma) = 3$. Let $\Gamma(u) = \bigcup_{i=1}^m B_i$ such that each $[B_i]$ is a connected component of $[\Gamma(u)]$. Since G_u is transitive on $\Gamma(u)$, all the $[B_i]$ are isomorphic. Since $\mathrm{girth}(\Gamma) = 3$, the size $|B_i| \geqslant 2$. Let $v_1, v_2 \in B_1$ and $w \in B_2$. Then (v_1,u,w) is a 2-geodesic and (v_1,u,v_2) is a 2-arc. Now G_{u,v_1} fixes B_1 setwise and so no element of G_{u,v_1} maps v_2 to w, and hence no element of G maps (v_1,u,v_2) to (v_1,u,w). Since Γ is $(G,2)$-geodesic transitive, it follows that (v_1,u,v_2) is not a 2-geodesic, that is, v_1, v_2 are adjacent. Thus $[B_1] \cong K_r$ where $r \geqslant 2$ and $mr = n$. Therefore, $[\Gamma(u)] \cong mK_r$, and (2) holds. This completes the proof.

4. Clique graphs and partial linear spaces

Let Γ be a connected $(G,2)$-geodesic transitive graph such that $[\Gamma(u)]$ is not connected. Then by Theorem 1.1, every such graph lies in the family $\mathcal{F}(m,r)$ for some $m \geqslant 2, r \geqslant 1$. In the first subsection, we initiate an investigation of the graphs in $\mathcal{F}(m,r)$ for all parameters $m \geqslant 1, r \geqslant 1$. Note that $\mathcal{F}(1,r)$ has just one member, namely the complete graph K_{r+1}, see Remark 1.5(a).

4.1. Clique graphs

We refer to a clique of size r as an r-clique.

Lemma 4.1. *Let $\Gamma \in \mathcal{F}(m,r)$ with $m \geqslant 1, r \geqslant 1$. Then the following statements hold.*

(1) *Every maximal clique contains exactly $r+1$ vertices.*
(2) *Any two maximal cliques of Γ have at most one common vertex. In particular, each edge of Γ lies in a unique maximal clique.*
(3) *Let $u \in V(\Gamma)$. Let $\Omega = \{[A_1],[A_2],\dots,[A_m]\}$ be the set of r-cliques of $[\Gamma(u)]$ and let $A_i' = A_i \cup \{u\}$. Then $[A_i'] \cong K_{r+1}$ and $[A_1'],[A_2'],\dots,[A_m']$ are all the maximal cliques that contain u.*
(4) *Suppose that $|V(\Gamma)| = n$. Then $|V(C(\Gamma))| = mn/(r+1)$. In particular, $|V(C(\Gamma))| = n$ if and only if $m = r+1$.*

Proof. Let $[\Delta]$ be a maximal clique and let u, v be distinct vertices in Δ. Then $[\Delta \setminus \{u\}]$ is a clique of $[\Gamma(u)]$ containing v, so $\Delta \setminus \{u\} \subseteq \{v\} \cup (\Gamma(u) \cap \Gamma(v))$, a set of size r. Thus $|\Delta| \leqslant r+1$. Further, by maximality of $[\Delta]$, $|\Delta| = r+1$ and $[\Delta \setminus \{u\}]$ is the unique maximal clique of $[\Gamma(u)]$ containing v. If Δ' is a maximal clique containing $\{u,v\}$, then this implies that $\Delta' = \Delta$, proving (2) as well as (1). This also implies part (3).

Finally, by (3), each vertex of Γ lies in exactly m maximal cliques, each of size $r+1$. It follows that Γ has $mn/(r+1)$ maximal cliques, that is, $|V(C(\Gamma))| = mn/(r+1)$. □

Corollary 4.2. *Let $\Gamma \in \mathcal{F}(m,r)$ where $m \geqslant 2, r \geqslant 1$. Then $C(\Gamma) \cong L(\Gamma)$ if and only if $r = 1$.*

Proof. If $r = 1$, then maximal cliques are edges by Lemma 4.1(1), so $C(\Gamma) = L(\Gamma)$. Conversely, suppose that $C(\Gamma) \cong L(\Gamma)$. By Lemma 4.1(4), $|V(C(\Gamma))| = m.|V(\Gamma)|/(r+1)$, and on the other hand, $|V(L(\Gamma))| = |E(\Gamma)| = mr.|V(\Gamma)|/2$. It follows that $r = 1$. □

The next lemma determines the local structure of $C(\Gamma)$ for $\Gamma \in \mathcal{F}(m,r)$.

Lemma 4.3. *Let $\Gamma \in \mathcal{F}(m,r)$ with $m \geqslant 2, r \geqslant 1$, and let $\Sigma = C(\Gamma)$. Then $[\Sigma(u')]_\Sigma \cong (r+1)K_{m-1}$ for each $u' \in V(\Sigma)$, so $\Sigma \in \mathcal{F}(r+1, m-1)$.*

Proof. By Lemma 4.1(1), every maximal clique of Γ contains exactly $r+1$ vertices. Let $u' \in \Sigma$, say $u' = [\Delta]_\Gamma$ with $\Delta = \{u_1,u_2,\dots,u_{r+1}\} \subseteq V(\Gamma)$.

Let $\Omega = \{\Delta_{u_i} \mid i = 1,2,\dots,r+1\}$ where $\Delta_{u_i} = \Gamma(u_i) \setminus \Delta$. Then since $[\Gamma(u_i)]_\Gamma \cong mK_r$ and $[\Delta]_\Gamma \cong K_{r+1}$, we have $[\Delta_{u_i}]_\Gamma \cong (m-1)K_r$. By Lemma 4.1(2), the edge $\{u_i,u_j\}$ lies in a unique maximal clique and this must be $[\Delta]_\Gamma$. It follows that $\Delta_{u_i} \cap \Delta_{u_j} = \emptyset$, that is, distinct elements of Ω are disjoint.

Now let $\overline{\Delta}_{u_i} = \{u_i\} \cup \Delta_{u_i}$. Since $\Delta_{u_i} \subseteq \Gamma(u_i)$ and $[\Delta_{u_i}]_\Gamma \cong (m-1)K_r$, it follows that $[\overline{\Delta}_{u_i}]_\Gamma$ consists of $m-1$ maximal cliques of Γ each pair of which intersects exactly in the vertex u_i. Thus $[\overline{\Delta}_{u_i}]_\Gamma$ corresponds to a set of $m-1$ pairwise adjacent vertices of Σ. We denote the set of $m-1$ maximal cliques of Γ in $[\overline{\Delta}_{u_i}]_\Gamma$ by A_i. Then $[A_i]_\Sigma \cong K_{m-1}$.

Since $\Delta \cap \overline{\Delta}_{u_i} = \{u_i\}$, each maximal clique of Γ in A_i is adjacent in Σ to $u' = [\Delta]_\Gamma$, so $[A_i]_\Sigma$ is a subgraph of $[\Sigma(u')]_\Sigma$. Since distinct elements of Ω are disjoint, distinct $\overline{\Delta}_{u_i}, \overline{\Delta}_{u_j}$ have no common vertices in Γ. Thus $[A_1 \cup A_2 \cup \cdots \cup A_{r+1}]_\Sigma \cong (r+1)K_{m-1}$ and is a subgraph of $[\Sigma(u')]_\Sigma$. Finally, if $u'' \in \Sigma(u')$ then $u'' = [\Delta'']_\Gamma$ is an $(r+1)$-clique of Γ containing a vertex of $u' = [\Delta]_\Gamma$, say u_i.

Since $u'' \neq u'$, we have $\Delta'' \neq \Delta$, so $\Delta'' \cap \Delta = \{u_i\}$ by Lemma 4.1(2). Hence $u'' \in A_i$. Thus $[\Sigma(u')]_\Sigma = [A_1 \cup A_2 \cup \cdots \cup A_{r+1}]_\Sigma \cong (r+1)K_{m-1}$. $\square$

The following lemma will be used in the next subsection.

Lemma 4.4. *Let $\Gamma \in \mathcal{F}(m,r)$ where $m \geqslant 2$, $r \geqslant 1$. Let $\Sigma = C(\Gamma)$ and $\Omega = \{[\Delta_1]_\Gamma, [\Delta_2]_\Gamma, \ldots, [\Delta_m]_\Gamma\} \subseteq V(\Sigma)$. Then the induced subgraph $[\Omega]_\Sigma \cong K_m$ if and only if $|\Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_m| = 1$ in Γ.*

Proof. Since each $[\Delta_i]_\Gamma$ is a vertex of Σ, it follows from Lemma 4.1(1) that $[\Delta_i]_\Gamma \cong K_{r+1}$. If $|\Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_m| = 1$, that is, $\Delta_1, \Delta_2, \ldots, \Delta_m$ have a unique common vertex in Γ, then by the definition of the clique graph, $[\Omega]_\Sigma \cong K_m$.

Conversely, suppose that $[\Omega]_\Sigma \cong K_m$. It follows from Lemma 4.1(2) that $|\Delta_i \cap \Delta_j| = 1$ in Γ for each $i \neq j$. If $m = 2$, the proof is completed, so assume that $m \geqslant 3$. Let $\{u_{12}\} = \Delta_1 \cap \Delta_2$, $\{u_{1i}\} = \Delta_1 \cap \Delta_i$ and $\{u_{2i}\} = \Delta_2 \cap \Delta_i$ for $i \in \{3, 4, \ldots, m\}$. If $u_{12} = u_{1i}$ for all $i \geqslant 3$, or if $u_{12} = u_{2i}$ for all $i \geqslant 3$, then $\{u_{12}\} = \Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_m$ as required. So assume this is not the case. We will derive a contradiction. Let e and f be minimal such that $u_{12} \neq u_{1e}$ and $u_{12} \neq u_{2f}$. Without loss of generality, assume that $3 \leqslant e \leqslant f$. First, assume that $e = f$. If $u_{1e} = u_{2e}$, then $u_{1e} \in \Delta_1 \cap \Delta_2$, and hence $\{u_{12}, u_{1e}\} \subseteq \Delta_1 \cap \Delta_2$, contradicting Lemma 4.1(2). Thus $u_{1e} \neq u_{2e}$, and u_{1e} and u_{2e} are adjacent as both lie in Δ_e. However, since $[\Gamma(u_{12})]_\Gamma \cong mK_r$, no vertices of $\Delta_1 \setminus \{u_{12}\}$ and $\Delta_2 \setminus \{u_{12}\}$ are adjacent, a contradiction. Thus $e < f$. Since $3 \leqslant e$, and by the minimality of f, it follows that $u_{12} = u_{2e}$. Since $u_{12} \neq u_{1e}$, it follows that $u_{1e} \neq u_{2e}$, and we have $\{u_{1e}, u_{2e}\} \subseteq \Delta_1 \cap \Delta_e$, contradicting $|\Delta_1 \cap \Delta_e| = 1$. $\square$

4.2. Partial linear spaces

In this section, we study partial linear spaces which correspond to graphs in $\mathcal{F}(m,r)$. This point of view helps us understand the graphs in $\mathcal{F}(m,r)$ deeper, and in particular, the correspondence between $\mathcal{F}(m,r)$ and $\mathcal{F}(r+1, m-1)$ becomes a simple application of the duality of partial linear spaces.

Lemma 4.5. *Let $\mathcal{S}$ be a partial linear space with incidence graph $\overline{\mathcal{S}}$. Then $\mathcal{S}$ has no triangles if and only if $\overline{\mathcal{S}}$ has girth at least* 8.

Proof. Suppose that $\mathcal{S}$ has no triangles. By [7, Lemma 5.1.1], $\overline{\mathcal{S}}$ has girth at least 6. Assume that $(p_1, \ell_1, p_2, \ell_2, p_3, \ell_3, p_1)$ is a 6-cycle of $\overline{\mathcal{S}}$ where $p_i \in \mathcal{P}$ and $\ell_i \in \mathcal{L}$. Then by definition, p_1, p_2 and p_3 form a triangle of $\mathcal{S}$, contradicting that $\mathcal{S}$ has no triangles. Hence $\overline{\mathcal{S}}$ has no 6-cycles, and so its girth is greater than 6. Since $\overline{\mathcal{S}}$ is bipartite, it follows that $\overline{\mathcal{S}}$ has girth at least 8.

Conversely, suppose that $\overline{\mathcal{S}}$ has girth at least 8. Assume that $\mathcal{S}$ has a triangle, say (p_1, p_2, p_3). Then there exist 3 distinct lines ℓ_1, ℓ_2 and ℓ_3 such that p_1, p_2 are incident with ℓ_1, p_2, p_3 are incident with ℓ_2 and p_3, p_1 are incident with ℓ_3. Hence $(p_1, \ell_1, p_2, \ell_2, p_3, \ell_3, p_1)$ is a 6-cycle of $\overline{\mathcal{S}}$, contradicting that $\overline{\mathcal{S}}$ has girth at least 8. Thus $\mathcal{S}$ has no triangles. $\square$

Definition 4.6. Let $\Gamma \in \mathcal{F}(m,r)$ with $m \geqslant 2$, $r \geqslant 1$. Let $\mathcal{P} = V(\Gamma)$ and $\mathcal{L} = V(C(\Gamma))$, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ be the set of pairs (p, ℓ) such that $p \in \ell$. Let $\mathcal{S}(\Gamma)$ be the triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, and let $\overline{\mathcal{S}(\Gamma)}$ be the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and edges all pairs (p, ℓ) such that $(p, \ell) \in \mathcal{I}$.

It turns out that $\mathcal{S}(\Gamma) = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a partial linear space of order $(m, r+1)$ with no triangles and $\overline{\mathcal{S}(\Gamma)}$ is its incidence graph.

Lemma 4.7. *Let $\Gamma \in \mathcal{F}(m,r)$ with $m \geqslant 2$, $r \geqslant 1$, let $\mathcal{S}(\Gamma)$ and $\overline{\mathcal{S}(\Gamma)}$ be as in Definition 4.6. Then $\mathcal{S}(\Gamma)$ is a partial linear space of order $(m, r+1)$ with no triangles. Further, $\overline{\mathcal{S}(\Gamma)}$ is the incidence graph of $\mathcal{S}(\Gamma)$, Γ is the $\mathcal{S}(\Gamma)$-point graph and $C(\Gamma)$ is the $\mathcal{S}(\Gamma)$-line graph.*

Proof. By Lemma 4.1(2), each edge of Γ lies in a unique maximal clique, and hence in $\mathcal{S}(\Gamma)$, two points are incident with at most one line and two lines are incident with at most one point. Also each point lies on $m \geqslant 2$ lines and each line contains $r+1 \geqslant 2$ points. Thus $\mathcal{S}(\Gamma) = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a partial linear space of order $(m, r+1)$, and by definition $\overline{\mathcal{S}(\Gamma)}$ is its incidence graph. Let $\mathrm{girth}(\overline{\mathcal{S}(\Gamma)}) = \mathrm{g}$. Then by [7, Lemma 5.1.1], $\mathrm{g} \geqslant 6$ and g is even. If $\mathrm{g} = 6$, then $\overline{\mathcal{S}(\Gamma)}$ has a 6-cycle $(p_1, \ell_1, p_2, \ell_2, p_3, \ell_3, p_1)$ such that $p_i \in \mathcal{P}$, $\ell_i \in \mathcal{L}$, and $p_j \neq p_k$, $\ell_j \neq \ell_k$ whenever $j \neq k$. Thus p_1, p_2, p_3 are pairwise adjacent in Γ. Since $\Gamma \in \mathcal{F}(m, r)$, it follows by Lemma 4.1(2) that for distinct j, k, there is only one maximal clique ℓ containing $\{p_j, p_k\}$. Since $\{p_1, p_2, p_3\}$ lies in some maximal clique, that clique must be ℓ. In particular, $\ell = \ell_1 = \ell_2 = \ell_3$, a contradiction. Hence $\mathrm{girth}(\overline{\mathcal{S}(\Gamma)}) \geqslant 8$, by Lemma 4.5, $\mathcal{S}(\Gamma)$ has no triangles.

Finally, from the definition of $\mathcal{S}(\Gamma)$ we know that Γ is the $\mathcal{S}(\Gamma)$-point graph and $C(\Gamma)$ is the $\mathcal{S}(\Gamma)$-line graph. □

Now we can prove the first assertion of Theorem 1.4.

Proof of Theorem 1.4(1). Suppose that $\Gamma \in \mathcal{F}(m, r)$, and let $\mathcal{S} = \mathcal{S}(\Gamma)$ be the triple as in Definition 4.6. Then by Lemma 4.7, $\mathcal{S}$ is a partial linear space of order $(m, r+1)$ with no triangles and Γ is the $\mathcal{S}$-point graph.

Conversely, let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space of order $(m, r+1)$ with no triangles. Let p_0 be a point of $\mathcal{S}$ and $\overline{\mathcal{S}}(p_0) = \{\ell_0, \ell_1, \ldots, \ell_{m-1}\}$ be the set of lines incident with p_0. Since any two lines are incident with at most one common point, it follows that $\overline{\mathcal{S}}(\ell_i) \cap \overline{\mathcal{S}}(\ell_j) = \{p_0\}$ for distinct $\ell_i, \ell_j \in \overline{\mathcal{S}}(p_0)$, and hence $|\overline{\mathcal{S}}_2(p_0) \cap \overline{\mathcal{S}}(\ell_i)| = r$ and $|\overline{\mathcal{S}}_2(p_0)| = mr$. Let $\overline{\mathcal{S}}_2(p_0) \cap \overline{\mathcal{S}}(\ell_i) = \{p_{i1}, p_{i2}, \ldots, p_{ir}\}$, for $0 \leqslant i \leqslant m-1$. If $s \neq t$ and p_{se}, p_{tf} with $e, f \in \{1, 2, \ldots, r\}$, lie on a common line, then (p_0, p_{se}, p_{tf}) is a triangle of $\mathcal{S}$, contradicting our assumption that $\mathcal{S}$ has no triangles. Thus p_{se} and p_{tf} lie on a common line if and only if $s = t$.

Let Γ be the $\mathcal{S}$-point graph. Then $\Gamma(p_0) = \overline{\mathcal{S}}_2(p_0)$ and $[\Gamma(p_0)]_\Gamma \cong mK_r$. Since p_0 is any point of $\mathcal{P}$, it follows that $\Gamma \in \mathcal{F}(m, r)$. □

If $\Gamma \in \mathcal{F}(m, r)$ for some $m \geqslant 2$, $r \geqslant 1$, then Lemma 4.3 shows that $C(\Gamma) \in \mathcal{F}(r+1, m-1)$. Thus we can define a partial linear space $\mathcal{S}(\Gamma)$ for Γ and also a partial linear space $\mathcal{S}(C(\Gamma))$ for $C(\Gamma)$ as in Definition 4.6 and Lemma 4.7. The following proposition shows that $\mathcal{S}(C(\Gamma))$ is isomorphic to the dual of $\mathcal{S}(\Gamma)$.

Proposition 4.8. *Let $\Gamma \in \mathcal{F}(m, r)$ for some $m \geqslant 2$, $r \geqslant 1$. Let $\mathcal{S} = \mathcal{S}(\Gamma)$ and $\mathcal{S}(C(\Gamma))$ be as in Definition 4.6 and let $\mathcal{S}^*$ be the dual of $\mathcal{S}$. Then the $\mathcal{S}^*$-point graph is isomorphic to $C(\Gamma)$ and $\mathcal{S}(C(\Gamma)) \cong \mathcal{S}^*$.*

Proof. By the definition of $\mathcal{S}$, the identity map $\sigma : V(C(\Gamma)) \mapsto V(C(\Gamma))$ given by $\sigma : x \mapsto x$ is an isomorphism between $C(\Gamma)$ and the $\mathcal{S}$-line graph. As $\mathcal{S}^*$ is the dual of $\mathcal{S}$, it follows that σ is also an isomorphism between $C(\Gamma)$ and the $\mathcal{S}^*$-point graph.

Let $\Sigma = C(\Gamma)$. Then by Lemma 4.3, $[\Sigma(u)]_\Sigma \cong (r+1)K_{m-1}$ for each $u \in V(\Sigma)$. Let $\mathcal{S}(\Sigma) = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ be as in Definition 4.6. Then σ defines an isomorphism between the point graphs of $\mathcal{S}(\Sigma)$ and $\mathcal{S}^*$.

Let $\Sigma' = C(\Sigma)$. Then again by Lemma 4.3, $[\Sigma'(u')]_{\Sigma'} \cong mK_r$ for each $u' \in V(\Sigma')$. Let $\ell' \in V(\Sigma')$. Then ℓ' is a maximal clique of Σ, say $\ell' = [\Delta]_\Sigma \cong K_m$ where $\Delta = \{[\Delta_1]_\Gamma, [\Delta_2]_\Gamma, \ldots, [\Delta_m]_\Gamma\} \subseteq V(\Sigma)$. By the definition of Σ, each $[\Delta_i]_\Gamma \cong K_{r+1}$. By Lemma 4.4, $[\Delta]_\Sigma \cong K_m$ implies that $|\Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_m| = 1$, say $\Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_m = \{\ell\}$.

We define $\phi : V(\Sigma') \mapsto V(\Gamma)$ by $\phi : \ell' \mapsto \ell$. Since ℓ is unique, ϕ is well defined. We claim that ϕ is a bijection: By Lemma 4.1(3), for any $v \in V(\Gamma)$, there are m maximal cliques $[B_1]_\Gamma, [B_2]_\Gamma, \ldots, [B_m]_\Gamma$ containing v. Let $B = \{[B_1]_\Gamma, [B_2]_\Gamma, \ldots, [B_m]_\Gamma\}$. Then $[B]_\Sigma \cong K_m$, and hence $[B]_\Sigma$ is a vertex of Σ', and we have $\phi([B]_\Sigma) = v$, that is, ϕ is surjective. Now suppose that $\phi(w_1) = \phi(w_2) = w$ where $w_1, w_2 \in V(\Sigma')$ and $w \in V(\Gamma)$. Then w_1, w_2 are maximal cliques of Σ, say $w_1 = [C]_\Sigma \cong K_m \cong [D]_\Sigma = w_2$ where $C = \{[C_1]_\Gamma, [C_2]_\Gamma, \ldots, [C_m]_\Gamma\} \subseteq V(\Sigma)$ and $D = \{[D_1]_\Gamma, [D_2]_\Gamma, \ldots, [D_m]_\Gamma\} \subseteq V(\Sigma)$. Thus $[C_i]_\Gamma \cong [D_i]_\Gamma \cong K_{r+1}$, and $C_1 \cap C_2 \cap \cdots \cap C_m = \{w\} = D_1 \cap D_2 \cap \cdots \cap D_m$. By

Lemma 4.1(3), $[C_1]_\Gamma, [C_2]_\Gamma, \ldots, [C_m]_\Gamma$ are all the maximal cliques containing w, and hence $\{[C_1]_\Gamma, [C_2]_\Gamma, \ldots, [C_m]_\Gamma\} = \{[D_1]_\Gamma, [D_2]_\Gamma, \ldots, [D_m]_\Gamma\}$, that is, $C = D$. Thus $w_1 = w_2$, and hence ϕ is injective. So ϕ is a bijection.

Now define $\psi : \mathcal{S}(\Sigma) \mapsto \mathcal{S}^*$ by $\psi : p \mapsto p^\sigma$, $\ell \mapsto \ell^\phi$ for $p \in \mathcal{P}_1$ and $\ell \in \mathcal{L}_1$. Choose $p \in \mathcal{P}_1$ and $\ell \in \mathcal{L}_1$. Then ℓ is a maximal clique of Σ, say $\ell = [\Delta]_\Sigma \cong K_m$ where $\Delta = \{[\Delta_1]_\Gamma, [\Delta_2]_\Gamma, \ldots, [\Delta_m]_\Gamma\} \subseteq V(\Sigma)$. By the definition of Σ, each $[\Delta_i]_\Gamma \cong K_{r+1}$. By the definition of ϕ, $\Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_m = \{\ell^\phi\}$.

Now p and ℓ are incident in $\mathcal{S}(\Sigma)$ if and only if $p \in \{[\Delta_1]_\Gamma, [\Delta_2]_\Gamma, \ldots, [\Delta_m]_\Gamma\}$, which holds if and only if ℓ^ϕ is a vertex of p. In turn this is true if and only if ℓ^ϕ, p^σ are incident in $\mathcal{S}^*$ as $p = p^\sigma$. Thus ψ is an isomorphism between $\mathcal{S}(\Sigma)$ and $\mathcal{S}^*$. □

Corollary 4.9. *Let $\Gamma \in \mathcal{F}(m, r)$ where $m \geqslant 2$, $r \geqslant 1$. Then $\Gamma \cong C(C(\Gamma))$.*

Proof. Let $\mathcal{S} = \mathcal{S}(\Gamma)$ be the partial linear space as in Definition 4.6, and let $\mathcal{P}$, $\mathcal{P}^*$, $\mathcal{P}^{**}$ be the point graphs of $\mathcal{S}, \mathcal{S}^*$ and $\mathcal{S}^{**}$, respectively. Then applying Theorem 1.4(1) and Proposition 4.8 to Γ, $C(\Gamma)$ and $C(C(\Gamma))$ we have $\Gamma \cong \mathcal{P}$, $C(\Gamma) \cong \mathcal{P}^*$, and $C(C(\Gamma)) \cong \mathcal{P}^{**}$, and also $\mathcal{S}^* \cong \mathcal{S}(C(\Gamma))$, $\mathcal{S}^{**} \cong \mathcal{S}(C(C(\Gamma)))$. However, $\mathcal{S}^{**} = \mathcal{S}$, and so $\mathcal{P}^{**} = \mathcal{P}$, that is, $C(C(\Gamma)) \cong \Gamma$. □

Remark 4.10. Let $\Gamma \in \mathcal{F}(m, r)$ where $m \geqslant 2$, $r \geqslant 1$, and let $\mathcal{S}(\Gamma) = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be as in Definition 4.6. Since Γ is the $\mathcal{S}(\Gamma)$-point graph and $C(\Gamma)$ is the $\mathcal{S}(\Gamma)$-line graph, it follows that $\Gamma \cong C(\Gamma)$ if and only if $\overline{\mathcal{S}(\Gamma)}$ is vertex transitive, or equivalently, if and only if $\mathcal{S}(\Gamma)$ is self-dual. In particular, $\Gamma \cong C(\Gamma)$ is only possible for $m = r + 1$ by Lemma 4.1(4).

A graph $\Gamma \in \mathcal{F}(2, 1)$ is just a cycle of length at least 4, so it does satisfy $\Gamma \cong C(\Gamma)$ and is 2-geodesic transitive. In the next smallest case, $\mathcal{F}(3, 2)$ contains 2-geodesic transitive graphs that are not isomorphic to their clique graph (the Hamming graph $H(3, 3)$ for instance), and also contains 2-geodesic transitive graphs that are isomorphic to their clique graph (the Kneser graph $KG_{6,2}$ for instance). Note that the partial linear space $\mathcal{S}(KG_{6,2})$ is the unique generalized quadrangle of order $(2, 2)$.

Now we can prove the second part of Theorem 1.4.

Proof of Theorem 1.4(2). Let $m \geqslant 2$, $r \geqslant 1$. By Theorem 1.4(1), $f_{m,r} : \Gamma \mapsto \mathcal{S}(\Gamma)$ defines a bijection from $\mathcal{F}(m, r)$ to the set $\Psi(m, r + 1)$ of all partial linear spaces $\mathcal{S}$ of order $(m, r + 1)$ without triangles, and Γ is the $\mathcal{S}(\Gamma)$-point graph. Let $f : \mathcal{S} \mapsto \mathcal{S}^*$ denote the natural bijection from $\Psi(m, r + 1)$ to $\Psi(r + 1, m)$, and define $\phi_{m,r}$ to be the composition $f_{m,r} \circ f \circ f^{-1}_{r+1,m-1}$. Then $\phi_{m,r}$ is a bijection from $\mathcal{F}(m, r)$ to $\mathcal{F}(r + 1, m - 1)$ which maps Γ to the $\mathcal{S}(\Gamma)^*$-point graph, which by Proposition 4.8 is isomorphic to $C(\Gamma)$. Thus $\phi_{m,r} : \Gamma \mapsto C(\Gamma)$. Moreover, $f_{r+1,m-1} \circ f^{-1} \circ f^{-1}_{m,r}$ maps $C(\Gamma)$ to the $\mathcal{S}(C(\Gamma))^*$-point graph, and the $\mathcal{S}(C(\Gamma))^*$-point graph is (isomorphic to) Γ since $\mathcal{S}(C(\Gamma))^* \cong \mathcal{S}(\Gamma)^{**} = \mathcal{S}(\Gamma)$ by Proposition 4.8. This second map is just $\phi_{r+1,m-1}$, and hence $\phi^{-1}_{m,r} = \phi_{r+1,m-1}$ in the sense described in Theorem 1.4. □

Finally, we prove Corollary 1.6 to determine when Γ is a line graph.

Proof of Corollary 1.6. In each of the cases (1)–(3), Γ is a line graph. Suppose conversely that Γ is a line graph, and let $\Gamma = L(\Sigma)$. Assume that $m \geqslant 3$. Since $[\Gamma(u)]_\Gamma \cong mK_r$, there exist vertices u_1, u_2, u_3 in $\Gamma(u)$ which are mutually non-adjacent. By the definition of a line graph, u, u_1, u_2, u_3 are edges of Σ, u_1, u_2, u_3 all have a common vertex with u and any two of them have no common vertex, which is impossible. Thus $m = 1$ or 2. Now, if $m = 1$, then $\Gamma \cong K_{r+1}$, and $\Gamma = L(\Sigma)$ where the only possibilities for Σ are $\Sigma \cong K_3$ or $K_{1,3}$ when $r = 2$, and $\Sigma = K_{1,r+1}$ for $r \neq 2$, so (1) or (2) holds. Now suppose that $m = 2$, so for each $u \in V(\Gamma)$, $[\Gamma(u)]_\Gamma \cong 2K_r$. It follows from Theorem 1.4 that for $\Sigma' = C(\Gamma)$, $[\Sigma'(u')]_{\Sigma'} \cong (r + 1)K_1$ for each $u' \in V(\Sigma')$ and girth$(\Sigma') \geqslant 4$. Thus by Remark 1.5(b), $C(\Sigma') = L(\Sigma')$, that is, $C(C(\Gamma)) = L(\Sigma')$. Again by Theorem 1.4, $\Gamma \cong C(C(\Gamma))$. Thus (3) holds. □

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